WESLEY COLLEGE
By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER ONE 2017
QUESTIONS OF REVIEW 3:
Vectors in 3 dimensions

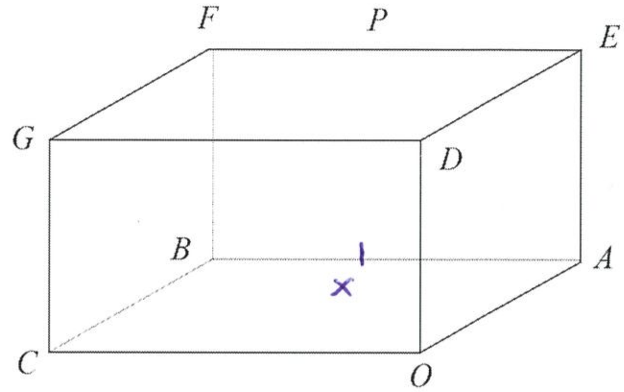
Name: SOLUTIONS

Thursday 11^{th} May | Time: 40 minutes | Mark /30

Calculator allowed.

1. [6 marks – 1, 1, 2 and 2]

The right rectangular prism shown has vertices $O(0,0,0)$, $A(3,0,0)$, $C(0,4,0)$ and $D(0,0,2)$.



Use appropriate vector methods to represent:

a) the position of point *G*

$(0,4,2)$ or $\vec{OG} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

b) the position of point *P*, the mid-point of *EF*

$\vec{OP} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$ or $(3,2,2)$

c) an equation for the line through points *C* and *P*

$\vec{r} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$ or equivalent such as $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}$

d) the angle the line *CP* makes with the base *OABC*

$\vec{CX} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$

$\text{angle}\left(\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} \right) = 29^{\circ}$ or 0.506^{R}

else $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$\Rightarrow \cos\theta = \dfrac{13}{\sqrt{13} \cdot \sqrt{17}} = 0.8745$

2. [4 marks – 2 and 2]

Use $\overrightarrow{OA} \otimes \overrightarrow{OB}$ and the vectors $\overrightarrow{OA} = 3i - 4j + k$ and $\overrightarrow{OB} = 5i + 4j - 2k$ to calculate

a) the area of ΔOAB

$$|A \times B| = |A|\,|B| \sin\theta .$$

$$\Rightarrow \text{area} = \frac{1}{2}\left|\vec{OA} \times \vec{OB}\right| = \frac{1}{2}\left|\begin{bmatrix}4\\11\\32\end{bmatrix}\right| = 17.0 \text{ units}^2$$

or $\frac{3\sqrt{129}}{2}$

b) $\angle AOB$

$$\sin\theta = \frac{34.07}{\sqrt{26}\cdot\sqrt{45}} = 0.9961$$

Spare

$$\therefore \theta = 85^\circ \text{ or } 1.483^R \text{ or their supplements}$$

$$\vec{OA} \bullet \vec{OB} < 0 \Rightarrow 2^{nd} \text{ quadrant} \Rightarrow \theta = 95^\circ \text{ or } 1.66^R$$

3. [9 marks – 1, 3, 2, 2 and 1]

a) Express the vector equation $\left|\vec{r} - \begin{bmatrix}1\\3\\1.5\end{bmatrix}\right| = \frac{9}{2}$ in Cartesian form

Sphere: $(x-1)^2 + (y-3)^2 + (z-1.5)^2 = \frac{81}{4}$

b) Calculate the point(s) of intersection of $\vec{r} = \begin{bmatrix}5\\1\\2\end{bmatrix} + \lambda\begin{bmatrix}8\\-4\\1\end{bmatrix}$ and $\left|\vec{r} - \begin{bmatrix}1\\3\\1.5\end{bmatrix}\right| = \frac{9}{2}$

Substitute: $\left|\begin{bmatrix}5+8\lambda-1\\1-4\lambda-3\\2+\lambda-1.5\end{bmatrix}\right| = \frac{9}{2}$

$$\Rightarrow (4+8\lambda)^2 + (-2-4\lambda)^2 + (0.5+\lambda)^2 = \frac{81}{4}$$

$$\Rightarrow \lambda = 0 \text{ or } -1$$

$\therefore$ points are $(5,1,2)$ & $(-3,5,1)$

c) Describe, by a suitable vector or algebraic equation, the locus of points that are equidistant from the points of intersection found in (b)

Plane through the midpt $(1, 3, 1.5) \perp \begin{bmatrix} 8 \\ -4 \\ +1 \end{bmatrix}$

with equation $\vec{r} \cdot \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix} = 8 - 12 + 1.5 = -2.5$ or $-\frac{5}{2}$

or $8x + 4y - z = -\frac{11}{2}$

d) Show that $\vec{r} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$ is part of a diameter of $\left| \vec{r} - \begin{bmatrix} 1 \\ 3 \\ 1.5 \end{bmatrix} \right| = \frac{9}{2}$

midpt of points of intersection is the centre $(1, 3, 1.5)$

$\Rightarrow \vec{r}$ is part of the sphere diameter.

e) Calculate the distance between any pair of parallel planes tangential to ~~both~~ opposite sides of

$$\left| \vec{r} - \begin{bmatrix} 1 \\ 3 \\ 1.5 \end{bmatrix} \right| = \frac{9}{2}$$

$d = 2r = 9$ units

4. [7 marks – 2, 2 and 3]

4 The vector $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$ is perpendicular to the plane Γ_1, which is itself parallel to both $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$.

a) Use scalar (dot) product calculations to set up equations sufficient to evaluate a and b

$$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 0 \quad \Rightarrow 2a + 3b - 1 = 0$$

$$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = 0 \quad \Rightarrow -a + b + 3 = 0$$

b) Use a vector (cross) product calculation to set up an equation to evaluate a and b

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = k \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \quad \Rightarrow \begin{array}{l} 9 + 1 = ka \\ 1 - 6 = kb \end{array}$$

but $6 - 1 = 5 = k$

c) Solve for a and b and hence develop a Cartesian equation for Γ_1, which passes through $(1,2,3)$

$$\left.\begin{array}{ll} 5a = 10 & a = 2 \\ 5b = -5 & b = -1 \end{array}\right\}$$

or simultaneously ...

$\therefore$ plane is $\vec{r} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = c = 3$

$\Rightarrow 2x - y + z = 3$

5. [4 marks]

The position vectors of three non-collinear points A, B and C, with respect to an origin O, are $\underset{\sim}{\mathbf{a}}$, $\underset{\sim}{\mathbf{b}}$ and $\underset{\sim}{\mathbf{c}}$ respectively.

O does not lie in the plane ABC.
The point Q with position vector $\underset{\sim}{\mathbf{q}} = \alpha\underset{\sim}{\mathbf{a}} + \beta\underset{\sim}{\mathbf{b}} + \gamma\underset{\sim}{\mathbf{c}}$ does lie in the plane ABC.

Show that $\alpha + \beta + \gamma = 1$

Plane is $\vec{r} = a + \lambda(a - b) + \mu(b - c)$ or similar.

$q = \alpha a + \beta b + \gamma c$.

$\Rightarrow \alpha = 1 + \lambda$

$\beta = -\lambda + \mu$

$\gamma = -\mu$

Adding: $\alpha + \beta + \gamma = 1$

Formulae:

Vectors

Magnitude:	$\lvert(a_1, a_2, a_3)\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$	
Dot product:	$\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$	
Triangle inequality:	$\lvert\mathbf{a}+\mathbf{b}\rvert \le \lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert$	
Vector equation of a line in space:	one point and the slope: two points A and B:	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$
Cartesian equations of a line in space:	$\dfrac{x-a_1}{b_1} = \dfrac{y-a_2}{b_2} = \dfrac{z-a_3}{b_3}$	
Parametric form of vector equation of a line in space:	$x = a_1 + \lambda b_1$......(1) $y = a_2 + \lambda b_2$......(2) $z = a_3 + \lambda b_3$......(3)	
Vector equation of a plane in space:	$\mathbf{r}\cdot\mathbf{n} = c$ or	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$
Cartesian equation of a plane:	$ax + by + cz = d$	

Vector cross product $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$ and $\lvert\vec{a}\times\vec{b}\rvert = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\sin\theta$

The sphere defined by $\lvert\vec{r} - \vec{a}\rvert = k$ has a centre with position vector $\vec{a}$ and radius k